BS401-MODULE – IV & V

NUMERICAL METHODS

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NOTES FOR NUMERICAL METHODS

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BISECTION METHOD

Bisection method is one of the bracketing methods. It is based on the "Intermediate value theorem"

The idea behind the method is that if f(x) ∈ C [a, b] and f(a).f(b)<0 then there exist a root "c ∈ (a,b)" such that "f(c)=0"

This method also known as BOLZANO METHOD (or) BINARY SECTON METHOD.

ALGORITHM

For a given continuous function f(x)

1. **Find a ,b such that f(a).f(b)<0 (this means there is a root "r ∈ (a,b)" such that f(r)=0**
2. **Let c = —— (mid-point)**
3. **If f(c)=0; done (lucky!)**
4. **Else; check if $f(c).f(a)$ or $f(c).f(b)$**
5. **Pick that interval [a, c] or [c, b] and repeat the procedure until stop criteria satisfied.**

STOP CRITERIA

1. **Interval small enough.**
2. **$|f(c_n)|$ almost zero**
3. **Maximum number of iteration reached**
4. **Any combination of previous ones**

CONVERGENCE CRITERIA

No. of iterations needed in the bisection method to achieve certain accuracy

Consider the interval $[a_0, b_0]$,, $c_0 = \frac{a_0+b_0}{2}$ and let $r \in (a_0,b_0)$ be a root then the error is $\in_0 = |r-c_0| \leq \frac{b_0-a_0}{2}$

Denote the further intervals as $[a_n,b_n]$ for iteration number "n" then

$$\in_n = |r-c_n| \leq \frac{b_n-a_n}{2} \leq \frac{b_0-a_0}{2^{n+1}} = \frac{\in_0}{2^n}$$

If the error tolerance is "$\in$" we require "$\in_n \leq \in$" then $\frac{b_0-a_0}{2^{n+1}} \leq \in$

After taking logarithm $\Longrightarrow$ $\log(b_0-a_0) - n\log 2 \leq \log(2\in)$

$$\Longrightarrow \frac{\log(b_0-a_0) - \log(2\in)}{\log 2} \leq n \Longrightarrow \frac{\log(b-a) - \log 2\in}{\log 2} \leq n \quad \text{(which is required)}$$

MERITS OF BISECTION METHOD

1. **The iteration using bisection method always produces a root, since the method brackets the root between two values.**
2. **As iterations are conducted, the length of the interval gets halved. So one can guarantee the convergence in case of the solution of the equation.**
3. **Bisection method is simple to program in a computer.**

DEMERITS OF BISECTION METHOD

1. **The convergence of bisection method is slow as it is simply based on halving the interval.**
2. **Cannot be applied over an interval where there is discontinuity.**
3. **Cannot be applied over an interval where the function takes always value of the same sign.**
4. **Method fails to determine complex roots (give only real roots)**
5. **If one of the initial guesses "a_0" or "b_0" is closer to the exact solution, it will take larger number of iterations to reach the root.**

EXAMPLE

Solve x^3-9x+1 for roots between x=2 and x=4

SOLUTION

X	2	4
f(x)	-9	29

Since f (2). f (4) <0 therefore root lies between 2 and 4

(1) $x_r = \frac{2+4}{2}$ = 3 so f(3) = 1 (+ve)

(2) For interval [2,3] ; $x_r = \frac{2+3}{2}$ = 2.5
f (2.5) = -5.875 (-ve)

(3) For interval [2.5,3]; x_r = (2.5+3)/2 = 2.75
f (2.75) = -2.9534 (-ve)

(4) For interval [2.75,3]; x_r = (2.75+3)/2 = 2.875
f (2.875) = -1.1113 (-ve)

(5) For interval [2.875,3]; x_r = (2.875+3)/2 = 2.9375
f (2.9375) = -0.0901 (-ve)

(6) For interval [2.9375,3]; x_r = (2.9375+3)/2 = 2.9688
f (2.9688) = +0.4471 (+ve)

(7) For interval [2.9375,2.9688]; x_r = (2.9375+2.9688)/2 = 2.9532
f (2.9532) = +0.1772 (+ve)

(8) For interval [2.9375,2.9532]; x_r = (2.9375+2.9532)/2 = 2.9453
f (2.9453) = 0.1772
Hence root is 2. 9453 because roots are repeated.

EXAMPLE

Use bisection method to find out the roots of the function describing to drag coefficient of parachutist given by

f(c) = $\frac{667.38}{c}$ [1-exp(-0.146843c)]-40 Where "c=12" to "c=16" perform at least two iterations.

SOLUTION

Given that f(c) = $\frac{667.38}{c}$ [1-exp(-0.146843c)]-40

X	12	13	14	15
f(x)	6.670	3.7286	1.5687	-0.4261

Since f (14). f (15) <0 therefore root lie between 14 and 15

X_r = $\frac{14+15}{2}$ = 14.5 So f(14.5) = 0.5537

Again f (14.5). f (15) <0 therefore root lie between 14.5 and 15

x_r = $\frac{14.5+15}{2}$ = 14.75 So f(14.75) = 0.0608 These are the required iterations

EXAMPLE

Explain why the equation $e^{-x} = x$ has a solution on the interval [0,1]. Use bisection to find the root to 4 decimal places. Can you prove that there are no other roots?

SOLUTION

If $f(x) = e^{-x} - x$, then f(0) = 1, f(1) = 1/e − 1 < 0, and hence a root is guaranteed by the Intermediate Value Theorem. Using Bisection, the value of the root is $x^? = .5671$.

Since $f^0(x) = -e^{-x} - 1 < 0$ for all x, the function is strictly decreasing, and so its graph can only cross the x axis at a single point, which is the root.

FALSE POSITION METHOD

This method also known as REGULA FALSI METHOD,, CHORD METHOD ,, LINEAR INTERPOLATION and method is one of the bracketing methods and based on intermediate value theorem.

This method is different from bisection method.

Like the bisection method we are not taking the mid-point of the given interval to determine the next interval and converge faster than bisection method.

ALGORITHM

Given a function f(x) continuous on an interval $[a_0,b_0]$ and satisfying $f(a_0).f(b_0)<0$ for all n = 0,1,2,3…………. then Use following formula to next root

$x_r = x_f - \frac{x_f - x_i}{f(x_f) - f(x_i)} f(x_f)$ **We can also use $x_r = x_{n+1}$,,, $x_f = x_n$,,, $x_i = x_{n-1}$**

STOPING CRITERIA

1. **Interval small enough.**
2. **$|f(c_n)|$ almost zero**
3. **Maximum number of iteration reached**
4. **Same answer.**
5. **Any combination of previous ones**

EXAMPLE

Using Regula Falsi method Solve x^3-9x+1 for roots between x=2 and x=4

SOLUTION

X	2	4
f(x)	-9	29

Since f(2).f(4)<0 therefore root lies between 2 and 4

Using formula

$$x_r = x_f - \frac{x_f - x_i}{f(x_f) - f(x_i)} f(x_f)$$

For interval [2,4] we have $x_r = 4 - \frac{4-2}{29-(-9)} \times 29 = 2.4737$

Which implies $f(2.4737) = -6.1263$ **(-ve)**

Similarly, other terms are given below

Interval	x_r	$F(x_r)$
[2.4737,4]	2.7399	-3.0905
[2.7399,4]	2.8613	-1.326
[2.8613,4]	2.9111	-0.5298
[2.9111,4]	2.9306	-0.2062
[2.9306,4]	2.9382	-0.0783
[2.9382,4]	2.9412	-0.0275
[2.9412,4]	2.9422	-0.0105
[2.9422,4]	2.9426	-0.0037
[2.9426,4]	2.9439	0.0183
[2.9426,2.9439]	2.9428	-0.0003
[2.9426,2.9439]	2.9428	-0.0003

EXAMPLE

Using Regula Falsi method to find root of equation "$logx - cosx = 0$" upto four decimal places, after 3 successive approximations.

SOLUTION

X	0	1	2
F(X)	-∞	-0.5403	1.1093

Since f(1).f(2)<0 therefore root lies between 1 and 2

Using formula

$$Xr = x_f - \frac{x_f - x_i}{f(x_f) - f(x_i)} f(x_f)$$

For interval [1,2] we have $x_r = 2 - \frac{2-1}{1.1093-(-0.5403)} \times 1.1093 = 1.3275$

Which implies f(2.4737)=0.0424(+ve)

Similarly, other terms are given below

Interval	x_r	$F(x_r)$
[1,1.3275]	1.3037	0.0013
[1,1.3037]	1.3030	0.0001

Hence the root is 1.3030

KEEP IN MIND

- **Calculate this equation in Radian mod**
- **If you have "log" then use "natural log". If you have "log_{10}" then use "simple log".**

GENERAL FORMULA FOR REGULA FALSI USING LINE EQUATION

Equation of line is

$$\frac{y - f(x_n)}{x - x_n} = \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

Put (x,0) i.e. y=0

$$\frac{-f(x_n)}{x - x_n} = \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

$$\frac{-f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n}$$

$$\frac{-(x_{n-1} - x_n)f(x_n)}{f(x_{n-1}) - f(x_n)} = x - x_n$$

$$x = x_n - \frac{(x_{n-1} - x_n)f(x_n)}{f(x_{n-1}) - f(x_n)}$$

Hence first approximation to the root of f(x) =0 is given by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}$$

We observe that f(x_{n-1}), f(x_{n+1}) are of opposite sign so, we can apply the above procedure to successive approximations.

NEWTON RAPHSON METHOD

Nature and Nature's laws lay hid in night:
God said, Let Newton be! And all was light.
Alexander Pope, 1727

The Newton Raphson method is a powerful technique for solving equations numerically. It is based on the idea of linear approximation. Usually converges much faster than the linearly convergent methods.

ALGORITHM

The steps of Newton Raphson method to find the root of an equation "f(x) =0" are

Evaluate $f'(x)$

Use an initial guess (value on which f(x) and $f''(x)$ becomes (+ve) of the roots "x_n" to estimate the new value of the root "x_{n+1}" as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots\ldots\ldots\ldots \textit{this value is known as Newton's iteration}$$

STOPING CRITERIA

1. **Find the absolute relative approximate error as** $|\epsilon_{\propto}| = |\frac{x_{n+1}-x_n}{x_{n+1}}| \times 100$
2. **Compare the absolute error with the pre-specified relative error tolerance "ϵ_s".**
3. **If | ϵ_a|> ϵ_s then go to next approximation. Else stop the algorithm.**
4. **Maximum number of iterations reached.**
5. **Repeated answer.**

CONVERGENCE CRITERIA

Newton method will generate a sequence of numbers (x_n) ; n$\geq$ 0, that converges to the zero "x_*" of "f" if

- **"f" is continuous.**
- **"x_*" is a simple zero of "f".**
- **"x_0" is close enough to "x_*"**

When the Generalized Newton Raphson method for solving equations is helpful?

To find the root of "f(x)=0" with multiplicity "p" the Generalized Newton formula is required.

What is the importance of Secant method over Newton Raphson method?

Newton Raphson method requires the evaluation of derivatives of the function and this is not always possible, particularly in the case of functions arising in practical problems.

In such situations Secant method helps to solve the equation with an approximation to the derivatives.

Why Newton Raphson method is called Method of Tangent?

In this method we draw tangent line to the point" $P_0(x_0,f(x_0))$". The (x,0) where this tangent line meets x-axis is 1^{st} approximation to the root.

Similarly, we obtained other approximations by tangent line. So, method also called Tangent method.

Difference between Newton Raphson method and Secant method.

Secant method needs two approximations x_0,x_1 to start, whereas Newton Raphson method just needs one approximation i.e. x_0

Newton Raphson method converges faster than Secant method.

Newton Raphson method is an Open method, how?

Newton Raphson method is an open method because initial guess of the root that is needed to get the iterative method started is a single point. While other open methods use two initial guesses of the root but they do not have to bracket the root.

INFLECTION POINT

For a function "f(x)" the point where the concavity changes from up-to-down or down-to-up is called its Inflection point.

e.g. $f(x) = (x-1)^3$ changes concavity at x=1,, Hence (1,0) is an Inflection point.

DRAWBACKS OF NEWTON'S RAPHSON METHOD

- **Method diverges at inflection point.**
- **For f(x)=0 Newton Raphson method reduce. So one must be avoid division by zero. Rather method not converges.**
- **Root jumping is another drawback.**
- **Results obtained from Newton Raphson method may oscillate about the Local Maximum or Minimum without converging on a root but converging on the Local Maximum or minimum.**

Eventually, it may lead to division by a number close to zero and may diverge.

- **The requirement of finding the value of the derivatives of f(x) at each approximation is either extremely difficult (if not possible) or time consuming.**

FORMULA DARIVATION FOR NR-METHOD

Given an equation "f(x) = 0" suppose "x_0" is an approximate root of "f(x) = 0"

Let $x_1 = x_0 + h \dots\dots\dots\dots\dots\dots (1)$ $\quad\quad$ $since\ \ x_1 - x_0 = h$

Where "h" is the small; exact root of f(x)=0

Then $f(x_1) = 0 = f(x_0 + h)$ $\quad\quad$ $since\ \ x_1 = x_0 + h$

By Taylor theorem

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) \dots\dots\dots\dots\dots\dots. = 0$$

Since "h" is small therefore neglecting higher terms we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

$$(1) \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots \ = \ \vdots - \quad \vdots$$

$$\vdots \ = \ \vdots - \quad \vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is required Newton's Raphson Formula.

EXAMPLE

Apply Newton's Raphson method for $cosx = xe^x\ at\ x_0 = 1$ correct to three decimal places.

SOLUTION

$f(x) = cosx - xe^x$

$f'(x) = -sinx - e^x - xe^x$

Using formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

at $x_0 = 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.653\quad (after\ solving)$

$f(x_1) = -0.460\ ;\ f'(x_1) = -3.783$

Similarly

n	x_n	$f(x_n)$	$f'(x_n)$
2	0.531	-0.041	-3.110
3	0.518	-0.001	-3.043
4	0.518	-0.001	-3.043

Hence root is "0.518"

REMARK

1. **If two are more roots are nearly equal, then method is not fastly convergent.**
2. **If root is very near to maximum or minimum value of the function at the point, NR-method fails.**

EXAMPLE

Apply Newton's Raphson method for $xlog_{10}x = 4.77$ correct to two decimal places.

SOLUTION

$$f(x) = xlog_{10}x - 4.77$$

$$f'(x) = log_{10}x + x\frac{1}{x}log_{10}e$$

$$f'(x) = log_{10}x + log_{10}e$$

$$f'(x) = log_{10}x + 0.4343 \qquad since\ e = 2.71828$$

$$f''(x) = \frac{1}{x}log_{10}e = \frac{0.4343}{x}$$

For interval

X	0	1	2	3	4	5	6	7
f(x)	-4.77	-4.77	-4.17	-3.34	-2.36	-1.28	-0.10	1.15

Root lies between 6 and 7 and let x_0 =7

Using formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Thus

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 6.10 \quad after\ solving$$

$$f(x_1) = 0.02\ ;\ f'(x_1) = 1.22$$

Similarly

n	x_n	$f(x_n)$	$f'(x_n)$
2	6.08	0.00	0.00

Hence root is "6.08"

GEOMETRICAL INTERPRETATION (GRAPHICS) OF NEWTON RAPHSON FORMULA

Suppose the graph of function "y=f(x)" crosses x-axis at "$\propto$" then "$x = \propto$" is the root of equation "$f(x) = 0$".

CONDITION

Choose "x_0" such that "$f(x)$" and f"(x) have same sign. If "$(x_0, f(x_0))$" is a point then slope of tangent at "$(x_0, f(x_0)) = m = \frac{dy}{dx}|_{(x_0, f(x_0))} = f'(x_0)$"

Now equation of tangent is

$$y - y_0 = m(x - x_0)$$

$$y - f(x_0) = f'(x_0)(x - x_0) \quad \text{... (i)}$$

Since $(x_1, f(x_1) = y_1 = 0)$ as we take x_1 as exact root

(i)⟹ $$0 - f(x_0) = f'(x_0)(x - x_0)$$

$$-\frac{f(x_0)}{f\prime(x0)} = x_1 - x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0)}$$

Which is first approximation to the root "$\propto$". If "P_1" is a point on the curve corresponding to "x_1" then tangent at "P_1" cuts x-axis at $P_1(x_2, 0)$ which is still closer to "$\propto$" than "x_1". Therefore "x_2" is a 2nd approximation to the root.

Continuing this process, we arrive at the root "$\propto$".

NEWTON SCHEME OF ITERATION FOR FINDING THE SQUARE ROOT OF POSITION NUMBER

The square root of "N" can be carried out as a root of the equation

$x = \sqrt{N} \Rightarrow x^2 = N \Rightarrow x^2 - N = 0$

Here $f(x) = x^2 - N$; $f(x_n) = x_n^2 - N$

$f'(x) = 2x$; $f'(x_n) = 2x_n$

Using Newton Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\Rightarrow x_{n+1} = x_n - \frac{(x_{n-N}^2)}{2x_n}$

$\Rightarrow x_{n+1} = \frac{1}{2}[x_n + \frac{N}{x_n}]$ **This is required formula.**

QUESTION

Evaluate $\sqrt{12}$ by Newton Raphson formula.

SOLUTION

Let $x = \sqrt{12} \Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0$

Here $f(x) = x^2 - 12$; $f'(x) = 2x$; $f''(x) = 2$

X	0	1	2	3	4
F(x)	-12	-11	-8	-3	4

Root lies between 3 and 4 and x_0 =4

Now using formula $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right] \Rightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{12}{x_n}\right]$ ……………………..(1)

For n=0 $x_1 = \frac{1}{2}\left[x_0 + \frac{12}{x_0}\right] \Rightarrow x_1 = \frac{1}{2}\left[4 + \frac{12}{4}\right]$=3.5

For n=2 $x_2 = \frac{1}{2}\left[x_1 + \frac{12}{x_1}\right] \Rightarrow x_2 = \frac{1}{2}\left[3.5 + \frac{12}{3.5}\right] = 3.4643$

Similarly $x_3 = 3.4641$ *and* $x_4 = 3.4641$

Hence $\sqrt{12} = 3.4641$

NEWTON SCHEME OF ITERATION FOR FINDING THE "pth" ROOT OF POSITION NUMBER "N"

Consider $x = N^{\frac{1}{p}} \quad \Rightarrow \quad x^p = N \Rightarrow x^p - N = 0$

Here $\quad f(x) = x^p - N \quad ; \quad f(x_n) = x_n^p - N$

$\quad f'(x) = px^{p-1} \quad ; \quad f'(x_n) = px_n^{p-1}$

Since by Newton Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \Rightarrow x_{n+1} = x_n - \frac{(x_n^p - N)}{(px_n^{p-1})} \quad \Rightarrow x_{n+1} = \frac{1}{px_n^{p-1}}[px_n^{p-1+1} - x_n^p + N]$$

$$x_{n+1} = \frac{1}{px_n^{p-1}}[(p-1)x_n^p + N] \quad \Rightarrow x_{n+1} = \frac{1}{p}\left[\frac{(p-1)x_n^p + N}{x_n^{p-1}}\right]$$ **Required formula for pth root.**

QUESTION

Obtain the cube root of 12 using Newton Raphson iteration.

SOLUTION

Consider $x = 12^{\frac{1}{3}} \quad \Rightarrow \quad x^3 = 12 \Rightarrow x^3 - 12 = 0$

Here $\quad f(x) = x^3 - 12$ **and** $f'(x) = 3x^2 \quad ; \quad f''(x) = 6x$

For interval

X	0	1	2	3
F(x)	-12	-11	-4	15

Root lies between 2 and 3 and x_0=3

Since by Newton Raphson formula for pth root.

$$x_{n+1} = \frac{1}{p}\left[\frac{(p-1)x_n^p + N}{x_n^{p-1}}\right] \quad \Rightarrow x_{n+1} = \frac{1}{3}\left[\frac{(3-1)x_n^3 + 12}{x_n^{3-1}}\right] = \frac{1}{3}\left[\frac{2x_n^3 + 12}{x_n^2}\right]$$

Put n=0 $\quad x_1 = \frac{1}{3}\left[\frac{2x_0^3 + 12}{x_0^2}\right] = \frac{1}{3}\left[\frac{2(3)^3 + 12}{(3)^2}\right] = 2.4444$

Similarly

$x_2 = 2.2990 \ , \ x_3 = 2.2895 \ , \ x_4 = 2.2894 \quad x_5 = 2.2894$

Hence $\quad \sqrt[3]{12} = 2.2894$

THE SOLUTION OF LINEAR SYSTEM OF EQUATIONS

A system of "m" linear equations in "n" unknowns "$x_1, x_2, x_3, \dots\dots\dots.., x_n$" is a set of the equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots\dots\dots\dots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots\dots\dots\dots a_{2n}x_n = b_2$$

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$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots\dots\dots\dots\dots a_{mn}x_n = b_m$$

Where the coefficients "a_{ik}" and "b_i" are given numbers.

The system is said to be homogeneous if all the "b_i" are zero. Otherwise it is said to be non-homogeneous.

SOLUTION OF LINEAR SYSTEM EQUATIONS

A solution of system is a set of numbers "$x_1, x_2, x_3, \dots\dots\dots.., x_n$" which satisfy all the "m" equations.

PIVOTING: **Changing the order of equations is called pivoting.**

We are interested in following types of Pivoting

1. **PARTIAL PIVOTING**
2. **TOTAL PIVOTING**

PARTIAL PIVOTING

In partial pivoting we interchange rows where pivotal element is zero.

In Partial Pivoting if the pivotal coefficient "a_{ii}" happens to be zero or near to zero, the i^{th} column elements are searched for the numerically largest element. Let the j^{th} row (j>i) contains this element, then we interchange the "i^{th}" equation with the "j^{th}" equation and proceed for elimination. This process is continued whenever pivotal coefficients become zero during elimination.

TOTAL PIVOTING

In Full (complete, total) pivoting we interchange rows as well as column.

In Total Pivoting we look for an absolutely largest coefficient in the entire system and start the elimination with the corresponding variable, using this coefficient as the pivotal coefficient (may change row and column). Similarly, in the further steps. It is more complicated than Partial Pivoting. Partial Pivoting is preferred for hand calculation.

Why is Pivoting important?

Because Pivoting made the difference between non-sense and a perfect result.

PIVOTAL COEFFICIENT

For elimination methods (Guass's Elimination, Guass's Jordan) the coefficient of the first unknown in the first equation is called Pivotal Coefficient.

BACK SUBSTITUTION

The analogous algorithm for upper triangular system "Ax=b" of the form

$$\begin{pmatrix} a_{11} & a_{12} \ldots\ldots\ldots.. a_{1n} \\ 0 & a_{22} \ldots\ldots\ldots.. a_{2n} \\ \vdots & \vdots \qquad\qquad \vdots \\ 0 & 0 \quad \cdots\cdots \quad a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Is called Back Substitution.

The solution "x_i" is computed by $x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \quad ; i = 1, 2, 3, \ldots\ldots\ldots n$

FORWARD SUBSTITUTION

The analogous algorithm for lower triangular system "Lx=b" of the form

$$\begin{pmatrix} l_{11} & 0 \quad \ldots\ldots.0 \\ l_{21} & l_{22} \ldots\ldots\ldots..0 \\ \vdots & \vdots \qquad\qquad \vdots \\ l_{n1} & l_{n2} \cdots\cdots \quad l_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Is called Forward Substitution.

The solution "x_i" is computed by $x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} x_j}{l_{ii}} \quad ; i = 1, 2, 3, \ldots\ldots\ldots n$

GUASS ELIMINATION METHOD

ALGORITHM

- **In the first stage, the given system of equations is reduced to an equivalent upper triangular form using elementary transformation.**
- **In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order "$x_n, x_{n-1}, \dots \dots \dots \dots x_2, x_1$"**

REMARK

Guass's Elimination method fails if any one of the Pivotal coefficient become zero. In such a situation, we rewrite the equation in a different order to avoid zero Pivotal coefficients.

QUESTION **Solve the following system of equations using Elimination Method.**

$$2x + 3y - z = 5$$

$$4x + 4y - 3z = 3$$

$$-2x + 3y - z = 1$$

SOLUTION **We can solve it by elimination of variables by making coefficients same.**

$$2x + 3y - z = 5 \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots (i)$$

$$4x + 4y - 3z = 3 \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots . (ii)$$

$$-2x + 3y - z = 1 \quad . . \dots \dots \dots \dots \dots \dots \dots \dots . . (iii)$$

Multiply (i) by 2 and subtracted by (ii) $2y + z = 7 \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (iv)$

Adding (i) and (iii) $6y - 2z = 6 \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (v)$

Now eliminating "y" Multiply (iv) by 3 then subtract from (v) $z = 3$

Using "z" in (iv) we get $y = 2$ **and Using "y", "z" in (i) we get** $x = 1$

Hence solution is $x = 1, y = 2, z = 3$

QUESTION

Solve the following system of equations by Guass's Elimination method with partial pivoting.

$$x + y + z = 7$$

$$3x + 3y + 4z = 24$$

$$2x + y + 3z = 16$$

SOLUTION

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 7 \\ 16 \end{bmatrix} \quad \sim R_{12} \quad \Rightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 16 \end{bmatrix} \quad \sim \frac{1}{3} R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 16 \end{bmatrix} \quad \sim R_2 - R_1 \Rightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 0 \end{bmatrix} \quad \sim R_3 - 2R_1$$

2nd row cannot be used as pivot row as a_{22} =0, So interchanging the 2nd and 3rd row we get

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \sim R_{23}$$

Using back substitution

$$-\frac{1}{3} z = -1 \quad \Rightarrow z = 3$$

$$-y + \frac{1}{3} z = 0 \quad \Rightarrow y = 3 \quad \therefore z = 3$$

$$x + y + \frac{4}{3} z = 8 \quad \Rightarrow x = 3 \quad \therefore y = 3 \; , \; z = 3$$

QUESTION

Solve the following system of equations using Guass's Elimination Method with partial pivoting.

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$

$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$

$$4x_1 + 5x_2 + 65x_3 + 2x_4 = 26$$

$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

SOLUTION

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ 26 \\ 21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix} \quad \sim R_{14}$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix} \quad \sim \frac{1}{9} R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 4 & 2 & 8 \\ 4 & 5 & 65 & 2 \\ 4 & 10 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 24 \\ 26 \\ 32 \end{bmatrix} \quad \sim R_{24}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 4 & 2 & 8 \\ 0 & \frac{29}{9} & \frac{85}{18} & 2 \\ 0 & \frac{74}{9} & \frac{29}{9} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 24 \\ 16.6668 \\ 22.668 \end{bmatrix} \quad \sim R_3 - 4R_1 \text{ and } \sim R_4 - 4R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 29/9 & 85/18 & 2 \\ 0 & 74/9 & 29/9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ 16.6668 \\ 22.668 \end{bmatrix} \quad \sim \frac{1}{4} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & ½ & 2 \\ 0 & 0 & 3.111 & -4.444 \\ 0 & 0 & -0.889 & 16.444 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ -2.6665 \\ -26.665 \end{bmatrix} \qquad \sim R_3 - \frac{29}{9}R_2 \text{ and } \sim R_4 - \frac{74}{9}R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 1 & -1.428 \\ 0 & 0 & 0 & 15.175 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ -0.857 \\ -27.427 \end{bmatrix} \qquad \sim \frac{R_3}{3.111} \text{ and } \sim R_4 + 0.889R_3$$

$\Rightarrow \quad 15.175x_4 = -27.427$

$\Rightarrow x_4 = -1.8074$

$\Rightarrow x_3 - 1.428x_4 = -0.857$

$\Rightarrow x_3 = -3.438 \quad \therefore x_4 = -1.8074$

$\Rightarrow x_2 + \frac{1}{2}x_3 + 2x_4 = 6$

$\Rightarrow x_2 = 11.3338 \quad \therefore x_4 = -1.8074 \;,\; x_3 = -3.438$

$\Rightarrow x_1 + \frac{4}{9}x_2 + \frac{4}{9}x_3 = 2.333$

$\Rightarrow x_1 = -1.1762 \quad \therefore x_2 = 11.3338 \;,\; x_3 = -3.438$

Hence required solutions are

$x_1 = -1.1762 \;,\; x_2 = 11.3338 \;,\; x_3 = -3.438 \;,\; x_4 = -1.8074$

UPPER TRIANGULATION MATRIX

A matrix having only zeros below the diagonal is called Upper Triangular matrix.

(OR)

A "$n \times n$" matrix "U" is upper triangular if its entries satisfy $u_{ij} = 0 \ for \ i > j$

i.e. $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

CROUTS REDUCTION METHOD

In linear Algebra this method factorizes a matrix as the product of a Lower Triangular matrix and an Upper Triangular matrix.

Method also named as Cholesky's reduction method, triangulation method, or LU-decomposition (Factorization)

ALGORITHM

For a given system of equations $\sum_1^n x_i = m \, ; m \in Z$

1. **Construct the matrix "A"**
2. **Use "A=LU" (without pivoting) and "PA=LU" (with pivoting) where "P" is the pivoting matrix and find "u_{ij}, l_{ij}"**
3. **Use formula "AX=B" where "X" is the matrix of variables and "B" is the matrix of solution of equations.**
4. **Replace "AX=B" by "LUX=B" and then put "UX=Z" i.e. "LZ=B"**
5. **Find the values of "$Z_{i's}$" then use "Z=UX" find "$X_{i's}$" ; i=1, 2, 3, …….n**

ADVANTAGE/LIMITATION (FAILURE)

1. **Cholesky's method widely used in Numerical Solution of Partial Differential Equation.**
2. **Popular for Computer Programming.**
3. **This method fails if $a_{ii} = 0$ in that case the system is Singular.**

QUESTION

Solve the following system of equations using Crout's Reduction Method

$$5x_1 - 2x_2 + x_3 = 4$$

$$7x_1 + x_2 - 5x_3 = 8$$

$$3x_1 + 7x_2 + 4x_3 = 10$$

ANSWER

Let $$A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

One of the diagonals of L or U must be 1

Step I….

$$[A] = [L][U]$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

After multiplication on R.H.S

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$$\Rightarrow l_{11} = 5, \quad l_{21} = 7, \quad l_{31} = 3$$

$$\Rightarrow l_{11}u_{12} = -2 \quad \Rightarrow 5u_{12} = -2 \quad \Rightarrow u_{12} = -2/5$$

$$\Rightarrow l_{11}u_{13} = 1 \quad \Rightarrow 5u_{13} = 1 \quad \Rightarrow u_{13} = 1/5$$

$$\Rightarrow l_{21}u_{12} + l_{22} = 1 \quad \Rightarrow 7(-2/5) + l_{22} = 1 \quad \Rightarrow l_{22} = 19/5$$

$$\Rightarrow l_{31}u_{12} + l_{32} = 7 \quad \Rightarrow 3(-2/5) + l_{32} = 7 \quad \Rightarrow l_{32} = 41/5$$

$$\Rightarrow l_{21}u_{13} + l_{22}u_{23} = -5 \quad \Rightarrow 7\left(\frac{1}{5}\right) + \left(\frac{19}{5}\right)u_{23} = -5 \Rightarrow u_{23} = -32/19$$

$$\Rightarrow l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4 \quad \Rightarrow 3\left(\frac{1}{5}\right) + \left(\frac{41}{5}\right)\left(\frac{-32}{5}\right) + l_{33} = 4 \quad \Rightarrow l_{33} = 327/19$$

Step II…. **Put** $[A][X] = [B] \quad \Rightarrow \quad [L][U][X] = [B]$

Put [U] [X] = [Z] $[L][Z] = [B]$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & 19/5 & 0 \\ 3 & 41/5 & 327/19 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

$$\Rightarrow 5z_1 = 4 \ \Rightarrow z_1 = 4/5 \Rightarrow 7z_1 + \frac{19}{5}z_2 = 8 \ \Rightarrow 7\left(\frac{4}{5}\right) + \frac{19}{5}z_2 = 8 \Rightarrow z_2 = 12/19$$

$$\Rightarrow 3z_1 + \frac{41}{5}z_2 + \frac{327}{19}z_3 = 10 \qquad \Rightarrow \quad 3\left(\frac{4}{5}\right) + \frac{41}{5}\left(\frac{12}{19}\right) + \frac{327}{19}z_3 = 10 \quad \Rightarrow z_3 = 46/327$$

Step III…. **Since** **[U][X] = [Z]**

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & 1 & -32/19 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 12/19 \\ 46/327 \end{bmatrix}$$

$$\Rightarrow x_3 = 46/327$$

$$\Rightarrow x_2 - \frac{32}{19}x_3 = \frac{12}{19} \ \Rightarrow x_2 - \frac{32}{19}\left(\frac{46}{327}\right) = \frac{12}{19} \qquad \Rightarrow x_2 = 284/327$$

$$\Rightarrow x_1 - \left(\frac{2}{5}\right)x_2 + \frac{1}{5}x_3 = \frac{4}{5} \Rightarrow x_1 - \left(\frac{2}{5}\right)\left(\frac{284}{327}\right) + \frac{1}{5}\left(\frac{46}{327}\right) = \frac{4}{5}$$

$$\Rightarrow x_1 = 366/327$$

Hence required solutions are

$$\Rightarrow x_1 = 366/327, \qquad x_2 = 284/327, \qquad x_3 = 46/327$$

$$z^{(4)} = \frac{71}{29} - \frac{8}{29}(1.4212) - \frac{3}{29}(1.0668) = \frac{71}{29} = 1.9451$$

$$\Rightarrow \quad \left(x^{(4)}, \quad y^{(4)}, \quad z^{(4)}\right) = (1.0529, \quad 1.3553, \quad 1.9451)$$

Put k = 4 for fifth iteration

$$x^{(51)} = \frac{95}{83} - \frac{11}{83}(1.3551) + \frac{4}{83}(1.9451) = 1.0587$$

$$y^{(5)} = \frac{104}{52} - \frac{7}{52}(1.0529) - \frac{13}{52}(1.9451) = 1.3726$$

$$z^{(5)} = \frac{71}{29} - \frac{8}{29}(1.3553) - \frac{3}{29}(1.0529) = 1.9655$$

$$\Rightarrow \quad \left(x^{(5)}, \quad y^{(5)}, \quad z^{(5)}\right) = (1.0587, \quad 1.3726, \quad 1.9655)$$

GUASS SEIDEL ITERATION METHOD

Guass's Seidel method is an improvement of Jacobi's method. This is also known as method of successive displacement.

ALGORITHM

In this method we can get the value of "x_1"from first equation and we get the value of "x_2" by using "x_1" in second equation and we get "x_3" by using "x_1" and "x_2" in third equation and so on.

ABOUT THE ALGORITHM

- **Need only one vector for both "x^k" and "x^{k+1}" save memory space.**
- **Not good for parallel computing.**
- **Converge a bit faster than Jacobi's.**

<u>How Jacobi method is accelerated to get Guass Seidel method for solving system of Linear Equations.</u>

In Jacobi method the (r+1)th approximation to the system $\sum_{j=1,j\neq i}^{n} a_{ij}x_j = b_i$ is given by $x_i^{r+1} = \frac{b_i}{a_{ii}} - \sum_{j=1,j\neq i}^{n} \frac{a_{ij}}{a_{ii}} x_j^r$; $r, j = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots n$ from which we can observe that no element of x_i^{r+1} replaces x_i^r entirely for next cycle of computations. However, this is done in Guass Seidel method. Hence called method of Successive displacement.

QUESTION: Find the solutions of the following system of equations using Guass Seidel method and perform the first five iterations.

$$x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_3 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$

$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$$

ANSWER

$$x_1 = 0.5 + 0.25x_2 + 0.25x_3$$

$$x_2 = 0.5 + 0.25x_1 + 0.25x_4$$

$$x_3 = 0.25 + 0.25x_1 + 0.25x_4$$

$$x_4 = 0.25 + 0.25x_2 + 0.25x_3$$

For first iteration using $(0, 0, 0, 0)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

For second iteration using $(0.5, 0.5, 0.25, 0.25)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0.25) + 0.25(0.25) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0.5) + 0.25(0.25) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0.5) + 0.25(0.25) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0.25) + 0.25(0.25) = 0.25$$

For third iteration using $(0, 0, 0, 0)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

For fourth iteration using $(0, 0, 0, 0)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

For fifth iteration using $(0, 0, 0, 0)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

COLLOCATION

Like the finite difference methods, the idea behind the collocation is to reduce the boundary value problem to a set of solvable algebraic equations.

However, instead of discretizing the differential equation by replacing derivative with finite differences, the solution is given a functional from whose parameters are fit by the method.

CRITERION OF APPROXIMATION

Some methods are as follows

i. **collocation** ii. **Osculation** iii. **Least square**

FORWARD DIFFERENCE OPERATOR ‘Δ’

We define forward difference operator as $\Delta y_i = y_{i+1} - y_i \quad i = 1, 2, \dots n-1$
Where y=f(x) (OR) $\Delta y_x = y_{x+h} - y_x$

For first order

Given function y=f(x) and a value of argument ‘x’ as x=a, a+h…… a+nh etc.
Where ‘h’ is the step size (increment) first order Forward Difference Operator is

$$\Delta f(a) = f(a+h) - f(a) \quad OR \quad \Delta y_i = y_{i+1} - y_i \quad \forall i = 1, 2, 3 \dots n-1$$

For Second Order

Let $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$
$= y_2 - 2y_1 + y_0$

For Third Order

$$\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta(y_2 - 2y_1 + y_0) = \Delta y_2 - 2\Delta y_1 + \Delta y_0$$
$$= (y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

$\Rightarrow \Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ **are called leading differences**

In General: $\Delta^n y_0 = y_n - {}^n_1C y_{n-1} + {}^n_2C y_{n-2} + \cdots + (-1)^n y_n$

Remark ${}^n_rC = \frac{n!}{r!(n-r)!}$ *and* ${}^n_0C = {}^n_nC = 1$ *and* ${}^n_1C = {}_{n-1}^{\ n}C = n$

CONSTRUCTION OF FORWARD DIFFERENCE TABLE (Also called Diagonal difference table)

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$\longrightarrow$ $\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\longrightarrow$ $\Delta^2 y_0$		
		$\longrightarrow$ $\Delta y_1 = y_2 - y_1$		$\longrightarrow$ $\Delta^3 y_0$	
x_2	y_2		$\longrightarrow$ $\Delta^2 y_1$		$\longrightarrow$ $\Delta^4 y_0$
		$\longrightarrow$ $\Delta y_2 = y_3 - y_2$		$\longrightarrow$ $\Delta^3 y_1$	
x_3	y_3		$\longrightarrow$ $\Delta^2 y_2$		
		$\longrightarrow$ $\Delta y_3 = y_4 - y_3$			
x_4	y_4				

QUESTION: **Construct forward difference Table for the following value of 'X' and 'Y'**

X	**0.1**	**0.3**	**0.5**	**0.7**	**0.9**	**1.1**	**1.3**
Y	**0.003**	**0.067**	**0.148**	**0.248**	**0.370**	**0.518**	**0.697**

SOLUTION

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.1	0.003						
		→ 0.064					
0.3	0.067		→ 0.017				
		→ 0.081		→ 0.002			
0.5	0.148		→ 0.019		→ 0.001		
		→ 0.100		→ 0.003		→ 0	
0.7	0.248		→ 0.022		→ 0.001		→ 0
		→ 0.122		→ 0.004		→ 0	
0.9	0.370		→ 0.026		→ 0.001		
		→ 0.148		→ 0.005			
1.1	0.518		→ 0.031				
		→ 0.179					
1.3	0.697						

INTERPOLATION

For a given table of values $(x_k, y_k)\ \forall\ K = 0, 1, 2, \ldots \ldots n.$ the process of estimating the values of "y=f(x)" for any intermediate values of "x = g(x)" is called "interpolation".

If g(x) is a Polynomial, Then the process is called "Polynomial" Interpolation.

ERROR OF APPROXIMATION

The deviation of g(x) from f(x) i.e. |f(x) – g (x)| is called Error of Approximation.

EXTRAPOLATION

The method of computing the values of 'y' for a given value of 'x' lying outside the table of values of 'x' is called Extrapolation.

REMARK

A function is said to interpolate a set of data points if it passes through those points.

INVERSE INTERPOLATION

Suppose $f \epsilon\ C\ [a, b]$, $f'(x)\ \neq\ o$ on [a, b] and f has non- zero 'p' in [a, b]
Let "$x_0, x_1\ \ldots \ldots \ldots . x_n$" be 'n+1' distinct numbers in [a, b] with $f\ (x_k)\ =\ y_k$ for each $k\ =\ 0, 1, 2\ \ldots . n.$
To approximate 'p' construct the interpolating polynomial of degree 'n' on the nodes "$y_0, y_1 \ldots\ldots\ldots y_n$" for "$f^{-1}$ "
Since "y_k=f (x_k)" and f (p) =0, it follows that f^{-1} (y_k) = X_k and p = f^{-1} (0).

"Using iterated interpolation to approximate $f^{-1}(0)$ is called iterated Inverse interpolation"

LINEAR INTERPOLATION FORMULA

$$f(x) = p_1(x) = f_0 + p(f_1 - f_0) = f_0 + p\Delta f_0$$

Where $x = x_0 + ph \quad \Rightarrow p = \frac{x - x_0}{h} \qquad 0\ \leq P \leq 1$

QUADRATIC INTERPOLATION FORMULA

$$f(x) = p_2(x) = f_0 + p\Delta f_0 + \frac{P(P-1)}{2}\Delta^2 f_0$$

Where $x = x_0 + ph \quad \Rightarrow p = \frac{x - x_0}{h} \qquad 0\ \leq P \leq 2$

NEWTON FORWARD DIFFERENCE INTERPOLATION FORMULA

Newton's Forward Difference Interpolation formula is

$$f(x) = P_n(x)$$
$$= f(x_0) + P\Delta f(x_0) + \frac{P(P-1)}{2!}\Delta^2 f(x_0) + \cdots + \frac{P(P-1)\cdots(P-n+1)}{n!}\Delta^n f(x_0)$$

Where $x = x_0 + ph, \quad P = \frac{x-x_0}{h} \quad And\ 0 \le p \le n$

DERIVATION:

Let $y = f(x), \quad x_0 = f(x_0) \quad And \quad x_n = x_0 + nh \quad \Rightarrow x = x_0 + ph$

$$f(x) = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p f(x_0) \qquad \therefore E = 1 + \Delta$$

$$= [1 + P\Delta + \frac{P(P-1)}{2!} + \cdots + \frac{P(P-1)\cdots(P-n+1)}{n!}]f(x_0)$$

$$f(x) = f(x_0) + P\Delta f(x_0) + \cdots + \frac{P(P-1)\cdots P-n+1)}{n!}f(x_0)$$

CONDITION FOR THIS METHOD

- **Values of 'x' must have equal distance i.e. equally spaced.**
- **Value on which we find the function check either it is near to start or end.**
- **If near to start, then use forward method.**
- **If near to end, then use backward method.**

QUESTION

Evaluate $f(15)$ given the following table of values

X	**:**	**10**	**20**	**30**	**40**	**50**
f(x)	**:**	**46**	**66**	**81**	**93**	**101**

SOLUTION

Here '15' nearest to starting point we use Newtown's Forward Difference Interpolation.

X	Y	ΔY	Δ^2Y	Δ^3Y	Δ^4Y
10	46				
		20			
20	66		-5		
		15		2	
30	81		-3		-3
		12		-1	
40	93		-4		
		8			
50	101				

$$f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\therefore x = x_0 + ph \Rightarrow 15 = 10 + P(10) \Rightarrow P = 0.5$$

$$f(15) = 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2!}(-5) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(2) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(-3)$$

$$\Rightarrow f(15) = 56.8672$$

NEWTONS'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

Newton's Backward Difference Interpolation formula is

$$y_x = f(x) \approx P_n(x)$$

$$= f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots\cdots + \frac{P(P+1)(P+2)\cdots\cdots(P+n-1)}{n!}\nabla^n f(x_n)$$

$$Where\ x = x_n + ph,\ \ p = \frac{x-x_n}{h};\ \ -n \le P \le 0$$

DERIVATION: Let $y = f(x), x_n = f(x_n)$ *and* $x = x_n + Ph$ Then

$$f(x_n + Ph) = E^P f(x_n) = (E^{-1})^{-P} f(x_n) = (1-\nabla)^{-P} f(x_n) \qquad \therefore E^{-1} = 1 - \nabla$$

Using binomial expansion $f(x) = \left[1 + P\nabla + \frac{P(P+1)}{2!}\nabla^2 + \frac{P(P+1)(P+2)}{3!}\nabla^3 + \cdots\right] f(x_n)$

$$f(x) = f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots\cdots$$

This is required <u>Newton's Gregory</u> Backward Difference Interpolation formula.

QUESTION: **For the following table of values estimate f(7.5)**

X	1	2	3	4	5	6	7	8
f(x)	1	8	27	64	125	216	343	512

SOLUTION

Since '7.5' is nearest to End of table, So We use Newton's Backward Interpolation.

X	Y	∇Y	$\nabla^2 Y$	$\nabla^3 Y$	$\nabla^4 Y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	243		42		
		169			
8	512				

Since $P = \frac{x - x_n}{h} \Longrightarrow P = \frac{7.5-8}{1} \Longrightarrow P = -0.5$

Now $y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$

$$y = 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2!}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(6)$$

$$y = 512 - 84.5 - 5.26 - 0.375 = f(x) = 421.875$$

LAGRANGE'S INTERPOLATION FORMULA

For points $x_0, x_1 \ldots \ldots \ldots \ldots . x_n$ define the cardinal Function

$l_0, l_1 \ldots \ldots \ldots \ldots . l_n \in P^n$ **(polynomial of n-degree)**

$$l_i(x_j) = \begin{cases} 1 & i=j \\ o & i \neq j \end{cases} \qquad \mathrm{i = o, 1, 2, \ldots . . n}$$

The Lagrange form of interpolation Polynomial is $p_n(x) = \sum_{i=0}^{n} l_i\,(x) y_i$

DERIVATION OF FORMULA

Let y=f (x) be a function which takes the values $y_0, y_1, y_2 \ldots \ldots \ldots y_n$ **so we will obtain an n-degree polynomial** $f(x) = a_0\, x^n + a_1 x^{n-1} + \ldots\ldots\ldots\ldots + a_n$

Now $(i) \ldots \ldots \ldots \ldots .$
$$\begin{cases} y = f(x) = a_0(x - x_1)(x - x_2) \cdots \cdots (x - x_n) \\ \quad + a_1(x - x_0)(x - x_2) \cdots \cdots (x - x_n) \\ \quad + a_2(x - x_0)(x - x_1)(x - x_3) \cdots \cdots (x - x_n) \\ \quad \vdots \\ \quad \vdots \\ \quad + a_n(x - x_0)(x - x_1) \cdots \cdots (x - x_{n-1}) \end{cases}$$

Now we find the constants $a_0, a_1, \cdots \cdots a_n$

Put x= x_0 **in (i)**

$$(i) \Rightarrow \begin{cases} y = f(x) = a_0(x_0 - x_1)(x_0 - x_2) \cdots \cdots (x_0 - x_n) \\ \quad + a_1(x_0 - x_0)(x_0 - x_2) \cdots \cdots (x_0 - x_n) \\ \quad + a_2(x_0 - x_0)(x_0 - x_1)(x_0 - x_3) \cdots \cdots (x_0 - x_n) \\ \quad \vdots \\ \quad \vdots \\ \quad + a_n(x_0 - x_0)(x_0 - x_1) \cdots \cdots (x_0 - x_{n-1}) \end{cases}$$

$$\Rightarrow \quad y_o = a_o(x_o - x_1)(x_o - x_2) \cdots \cdots (x_o - x_n)$$

$$\Rightarrow \quad a_0 = y_1 \div [(x_0 - x_1)(x_0 - x_2) \cdots \cdots (x_0 - x_1)]$$

Now Put x= $x_1\ in$

$$y_1 = f(x_1) = a_1(x_1 - x_o)(x_1 - x_2) \cdots \cdots (x_1 - x_n)$$

$$\Rightarrow \quad a_1 = y_1 \div [(x_n - x_0)(x_n - x_2) \cdots \cdots (x_n - x_{n-1})]$$

Similarly

$$a_n = y_n \div [(x_n - x_0)(x_n - x_1)\cdots\cdots(x_n - x_{n-1})]$$

Putting all the values in (i) we get

$$y = f(x) = y_0 \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} + y_1 \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} + \ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots. + y_n \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}$$

$$\Longrightarrow y = f(x) = l_0 y_0 + l_1 y_1 + l_2 y_2 + \cdots\cdots + l_n y_n = \sum_{k=0}^{n} l_k y_k$$

Where $l_k(x) = \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{k-1})(x-x_{k+1})\ldots\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots\ldots(x_k-x_{k-1})(x-x_{k+1})\ldots\ldots(x_k-x_n)}$

ALTERNATIVELY DEFINE

$$\pi(x) = (x - x_0)(x - x_1)\ldots\ldots(x - x_n)$$

Then $\pi'(x) = (1-0)[(x - x_1)(x - x_2)\ldots\ldots(x - x_n)]$

$$+(1-0)[(x - x_0)(x - x_2)(x - x_3)\ldots\ldots(x - x_n)]\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(1-0)[(x - x_0)(x - x_1)(x - x_2)\ldots\ldots(x - x_{n-1})]$$

$$\pi'(x_k) = (x_k - x_0)(x_k - x_1)\ldots\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots\ldots(x_k - x_n)$$

$$l_k(x) = \frac{(x-x_k)}{(x-x_k)} \cdot \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{k-1})(x-x_{k+1})\ldots\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots\ldots(x_k-x_{k-1})(x-x_{k+1})\ldots\ldots(x_k-x_n)}$$

Then $l_k(x) = \frac{\pi(x)}{(x-x_k)\pi'(x)}$

CONVERGENCE CRITERIA

Assume a triangular array of interpolation nodes $x_i = {x_i}^{(n)}$ exactly '$n+1$' distinct nodes for "$n = 0, 1, 2 \ldots\ldots\ldots i$"

$x_0^{(0)}$

$x_0^{(1)}$ $x_1^{(1)}$

$x_0^{(2)}$ $x_1^{(2)}$ $x_2^{(2)}$

$x_0^{(n)}$ $x_1^{(n)}$ $x_2^{(n)}$ $\ldots\ldots\ldots x_n^{(n)}$

QUESTION

Find langrage's Interpolation polynomial fitting The points $y(1) = -3$,

$y(3) = 0, y(4) = 30, y(6) = 132,$ **Hence find $y(5) = ?$**

X:	x_0=1	x_1=3	x_2=4	x_3=6
Y:	-3	0	30	132

ANSWER

Since $y(x) = l_o y_o + l_1 y_1 + l_2 y_2 + l_3 y_3$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_o + \frac{(x-x_o)(x-x_2)(x-x_3)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_o)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_o)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

By putting values, we get

$$y(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0) + \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

$$y(x) = \frac{1}{2}[-x^3 + 27x^2 - 92x + 60]$$

Put $x = 5$ **to get** $y(5)$

$$y(5) = \frac{1}{2}[-5^3 + 27(5^2) - 92(5) + 60] \Longrightarrow \textbf{Y (5) =75}$$

DIVIDED DIFFRENCE

Assume that for a given value of $(x_1, y_1)(x_{2,} y_2) \ldots . (x_n, y_n)$

$$y[x_0] = y(x_0) = y_0 \rightarrow y\ at\ x_0$$

Then the first order divided Difference is defined as

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_o} \quad , y[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1} = a_1$$

The 2nd Order Difference is $y[x_{0,} x_1, x_2] = \frac{y[x_1, x_2] - y[x_{o,} x_1]}{x_{2,} x_0} = a_2$

Similarly $y[x_{0,} x_{1,} x_2 \ldots \ldots x_n] = \frac{y[x_1, x_{2,} x_3 \ldots\ldots x_n] - y[x_{o,} x_1 \ldots\ldots x_{n-1}]}{x_n - x_o} = a_n$

NUMARICAL DIFFERENTIATION

The problem of numerical differentiation is the determination of approximate values the derivatives of a function 'f' at a given point.

DIFFERENTIATION USING DIFFERENCE OPERATORS

We assume that the function $y = f(x)$ is given for the equally spaced 'x' values $x_n = x_0 + nh$ for $n = 0, 1, 2, \ldots \ldots \ldots ..$ to find the darivatives of such a tabular function, we proceed as follows;

USING FORWARD DIFFERENCE OPERATOR $'\Delta'$

Since $hD = \log E = \log(1+\Delta)$ $\quad \therefore E = (1+\Delta)$

$\Rightarrow D = \frac{1}{h}[\log(1+\Delta)]$ **Where D is differential operator.**

$$\Rightarrow D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right] \quad \ldots \ldots \ldots \ldots \ldots \ldots (i) \quad \textit{using Maclaurin series}$$

Therefore

$$D\, f(x_0) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right] f(x_0) = f'(x_0)$$

$$D\, f(x_0) = f'(x_0) = \frac{1}{h}\left[\Delta f(x_0) - \frac{\Delta^2}{2} f(x_0) + \frac{\Delta^3}{3} f(x_0) - \frac{\Delta^4}{4} f(x_0) + \cdots \ldots \ldots \ldots \ldots .\right]$$

$$Dy_0 = y_0' = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2}{2} y_0 + \frac{\Delta^3}{3} y_0 - \frac{\Delta^4}{4} y_0 \ldots \ldots \ldots ..\right]$$

Similarly, for second derivative

$$(i) \Longrightarrow \qquad D^2 = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} \ldots \ldots \ldots ..\right]^2$$

$$D^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots \ldots \ldots ..\right] \quad \textbf{After solving}$$

$$D^2 y_0 = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots \ldots \ldots ..\right] = y_0''$$

USING BACKWARD DIFFERENCE OPERATOR "∇"

Since $hD = \log E = \log(E^{-1})^{-1} = -1 log\, E^{-1} = -1\log(1-\nabla)$

Since $log(1-\nabla) = -\nabla - \frac{\nabla^2}{2} - \frac{\nabla^3}{2} - \cdots \ldots \ldots \ldots \ldots \ldots$ **therefore**

$$\Rightarrow D = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \frac{\nabla^4}{4} + \cdots\right] \quad \ldots \ldots \ldots \ldots \ldots \ldots (i)$$

Now $D\, f(x_n) = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \frac{\nabla^4}{4} + \cdots\right] f(x_n) = f'(x_n)$

$$D\, f(x_n) = f'(x_n) = \frac{1}{h}\left[\nabla f(x_n) + \frac{\nabla^2}{2} f(x_n) + \frac{\nabla^3}{3} f(x_n) - \frac{\nabla^4}{4} f(x_n) + \cdots\right]$$

$$Dy_n = y_n' = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2}{2} y_n + \frac{\nabla^3}{3} y_n - \frac{\nabla^4}{4} y_n + \cdots\right]$$

Similarly, for second derivative squaring (i) we get

$$(i) \Longrightarrow \qquad D^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \cdots \ldots \ldots \ldots \ldots\right]$$

$$D^2 y_n = y_n'' = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots \ldots \ldots . \ldots\right]$$

<u>TO COMPUTE DARIVATIVE OF A TABULAR FUNCTION AT POINT NOT FOUND IN THE TABLE</u>

Since

$y(x_n + ph) =$

$$f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots \cdots + \frac{P(P+1)(P+2)\cdots\cdots(P+n-1)}{n!}\nabla^n f(x_n) \quad \ldots \ldots \ldots \ldots (i)$$

$$Where \quad x = x_n + ph \qquad \Longrightarrow \; p = \frac{x - x_n}{h};\; -n \le P \le 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (ii)$$

$$(i) \Longrightarrow \qquad y = f(x) = f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \ldots iii)$$

Differentiate with respect to 'x' and using (i) & (ii)

$$y' = \frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx} = \frac{d}{dp}\left[f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots \ldots \ldots \ldots\right]\frac{d}{dx}\left(\frac{x - x_n}{h}\right)$$

$$y' = \frac{d}{dp}\left[0 + \nabla f(x_n) + \frac{(2P+1)}{2}\nabla^2 f(x_n) + \cdots \ldots \ldots \ldots\right]\left(\frac{1-0}{h}\right)$$

$$y' = \frac{1}{h}\left[\nabla f(x_n) + \frac{(2P+1)}{2}\nabla^2 f(x_n) + \left(\frac{3P^2+6P+2}{6}\right)\nabla^3 f(x_n) + \left(\frac{4P^3+18P^2+22P+6}{24}\right)\nabla^4 f(x_n) \dots\dots\dots\dots\right] \quad \dots\dots (iv)$$

Differentiate y' with respect to 'x'

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dp}.\frac{dp}{dx} = \frac{1}{h}\left[\nabla^2 f(x_n) + (P+1)\nabla^3 f(x_n) + \left(\frac{6P^2+18P+11}{12}\right)\nabla^4 f(x_n) \dots\dots\dots\dots\right] \quad \dots\dots (v)$$

Equation (iv)& (v) are Newton's backward interpolation formulae which can be used to compute 1st and 2nd derivatives of a tabular function near the end of table similarly

Expression of Newton's forward interpolation formulae can be derived to compute the 1st, 2nd and higher order derivatives near the beginning of table of values.

DIFFERENTIATION USING CENTRAL DIFFERENCE OPERATOR (σ)

Since $\sigma = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Since $hD = \log E$ and $E = e$ **therefore** $\sigma = e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}$

Also as $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$ **therefore** $\sigma = 2\sin\left(\frac{hD}{2}\right)$

$$\Rightarrow \frac{\sigma}{2} = \sinh\left(\frac{hD}{2}\right) \quad \Rightarrow \sinh^{-1}\left(\frac{\sigma}{2}\right) = \left(\frac{hD}{2}\right) \quad \Rightarrow D = \frac{2}{h}\sinh^{-1}\frac{\sigma}{2}$$

Since by Maclaurin series

$$\sinh^{-1}(x) = x - \frac{1}{2}\left(\frac{x^3}{3}\right) + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots\dots\dots\dots$$

$$\Rightarrow D = \frac{2}{h}\left[\frac{\sigma}{2} - \frac{1}{2}\left(\frac{\left(\frac{\sigma}{2}\right)^3}{3}\right) + \frac{1.3}{2.4}\frac{\left(\frac{\sigma}{2}\right)^5}{5} - \frac{1.3.5}{2.4.6}\frac{\left(\frac{\sigma}{2}\right)^7}{7} + \cdots\right]$$

$$D = \frac{1}{h}\left[\sigma - \frac{\sigma^3}{24} + \frac{3\sigma^5}{640}\dots\dots\dots\dots\right] \quad \dots\dots\dots\dots (i)$$

NUMERICAL INTEGRATION

The process of producing a numerical value for the defining integral $\int_a^b f(x)dx$ is called Numerical Integration. Integration is the process of measuring the Area under a function plotted on a graph. Numerical Integration is the study of how the numerical value of an integral can be found.

Also called Numerical Quadrature if $\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i)$ which refers to finding a square whose area is the same as the area under the curve.

A GENERAL FORMULA FOR SOLVING NUMERICAL INTEGRATION

This formula is also called a general quadrature formula.

Suppose f(x) is given for equidistant value of 'x' say a=x_0, x_0+h,x_0+2h …. x_0+nh = b

Let the range of integration (a,b) is divided into 'n' equal parts each of width 'h' so that "b-a=nh".

By using fundamental theorem of numerical analysis It has been proved the general quadrature formula which is as follows

$$I = h\left[nf(x_0) + \frac{n^2}{2}\Delta f(x_0) + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f(x_0)}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f(x_0)}{3!} + \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11}{3}n^3 - 3n^2\right)\frac{\Delta^4 f(x_0)}{4!} + \cdots \ldots \ldots \ldots \ldots \ldots .. + up\ to\ (n+1) terms\right]$$

Bu putting n into different values various formulae is used to solve numerical integration.

That are Trapezoidal Rule, Simpson's 1/3, Simpson's 3/8, Boole's, Weddle's etc.

IMPORTANCE: **Numerical integration is useful when**

- **Function cannot be integrated analytically.**
- **Function is defined by a table of values.**
- **Function can be integrated analytically but resulting expression is so complicated.**

COMPOSITE (MODIFIED) NUMERICAL INTEGRATION

Trapezoidal and Simpson's rules are limited to operating on a single interval. Of course, since definite integrals are additive over subinterval, we can evaluate an integral by dividing the interval up into several subintervals, applying the rule separately on each one and then totaling up. This strategy is called Composite Numerical Integration.

TRAPEZOIDAL RULE

Rule is based on approximating $f(x)$ by a piecewise linear polynomial that interpolates $f(x)$ at the nodes "$x_0, x_1, \dots \dots \dots x_n$"

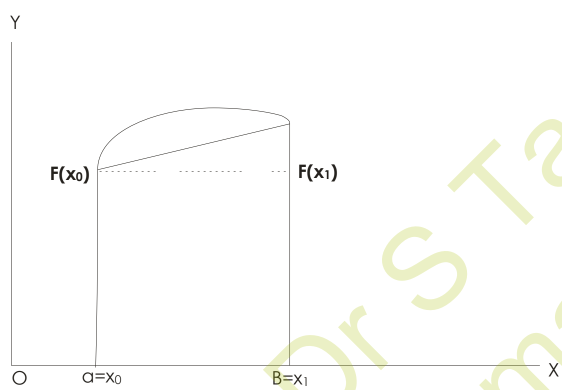
Trapezoidal Rule defined as follows

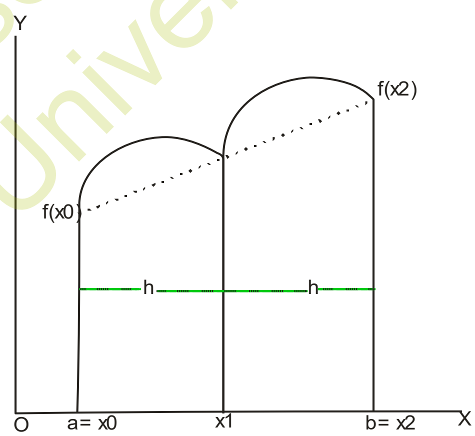
$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(y_0 + y_1) - \frac{h^3}{12}y''(a)$ **And this is called <u>Elementary Trapezoidal Rule</u>.**

Composite form of Trapezoidal Rule is $\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots \dots . + y_{n-1}) + y_n]$

DARIVATION (1st METHOD)

Consider a curve $y = f(x)$ bounded by $x_0 = a$ and $x_1 = b$ we have to find $\int_a^b f(x)dx$ i.e. Area under the curve $y = f(x)$ then for one Trapezium under the area i.e. n = 1





$$\int_a^b f(x)dx = Area\ of\ Trapezium = \frac{sum\ of\ parallel\ sides}{2} \times perpendicular$$

$$\int_a^b f(x)dx = \frac{f(x_0)+f(x_1)}{2} \times h = \frac{h}{2}[f(x_0) + f(x_1)]$$

For two trapeziums i. e. n = 2

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] = \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$$

For n = 3 $\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \frac{h}{2}[f(x_2) + f(x_3)]$

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2[f(x_1) + f(x_2)] + f(x_3)]$$

In general for n – trapezium the points will be "$x_0, x_1, \dots \dots ... x_n$" and function will be "$y_0, y_1, \dots \dots ... y_n$"

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2[f(x_1) + f(x_2) + \cdots \dots \dots \dots + f(x_{n-1})] + f(x_n)]$$

$$\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots \dots \dots \dots + y_{n-1}) + y_n]$$

Trapezium rule is valid for n (number of trapezium) is even or odd.

The accuracy will be increase if number of trapezium will be increased OR step size will be decreased mean number of step size will be increased.

DARIVATION (2nd METHOD)

Define y = f(x) in an interval $[a, b] = [x_0, x_n]$ then

$$\int_{x_0}^{x_0} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots \dots \dots \dots \dots .. + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$\int_{x_0}^{x_0} f(x)dx = \left[\frac{h}{2}(y_0 + y_1)\right] + \left[\frac{h}{2}(y_1 + y_2)\right] + \cdots \dots \dots \dots .. + \left[\frac{h}{2}(y_{n-1} + y_n)\right] + \epsilon_n$$

Where $\epsilon_n = -\frac{h^3}{12}[y''(a_1) + y''(a_2) + \cdots \dots \dots \dots \dots + y''(a_n)]$ **is global error.**

$\Rightarrow \epsilon_n = -\frac{h^3}{12}[ny''(a)]$

Therefore $\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots \dots \dots \dots + y_{n-1}) + y_n]$ **Where** $a = x_0 \; and \; b = x_n$

REMEMBER: **The maximum incurred in approximate value obtained by Trapezoidal Rule is nearly equal to** $\frac{(b-a)^3 M}{12n^2}$ **where** $M = max|f''(x)| \quad on \; [a, b]$

EXAMPLE: **Evaluate** $I = \int_0^1 \frac{1}{1+x^2} \mathrm{dx}$ **using Trapezoidal Rule when** $h = ¼$

SOLUTION

X	0	1/4	1/2	3/4	1
F(x)	1	0.9412	0.8000	0.6400	0.5000

Since by Trapezoidal Rule $\int_0^1 \frac{1}{1+x^2} \mathrm{dx} = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = 0.7828$

SIMPSON'S $\left(\frac{1}{3}\right)$ RULE

Rule is based on approximating f(x) by a Quadratic Polynomial that interpolate f(x) at x_{i-1}, x_i *and* x_{i+1}

Simpson's Rule is defined as for simple case $\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] - \frac{h^5}{90} y^{iv}(\S)$

While in composite form it is defined as

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots \ldots \ldots .. + y_{2N-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots + y_{2N-2}) + y_N]$$

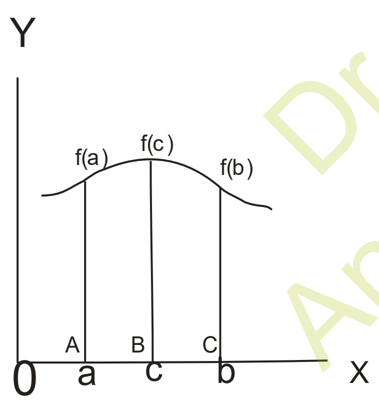
Global error for Simpson's Rule is defined as $\in = -\frac{x_{2N} - x_0}{180} h^4 y^{iv}(\S) = O(h^4)$

REMARK

In Simpson Rule number of trapezium must of Even and number of points must of Odd.

DERIVATION OF SIMPSON'S $\left(\frac{1}{3}\right)$ RULE (1st method)

Consider a curve bounded by x = a and x = b and let 'c' is the mid-point between a and b such that $a << b$ we have to find $\int_a^b f(x)dx$ i.e. Area under the curve.



Consider $X = C + Y \ldots \ldots \ldots \ldots ..(i) \Rightarrow \quad dx = dy$

Now $c = OB = OA + AB \Rightarrow c = a + h \Rightarrow a = c - h$

$b = OC = OB + BC \Rightarrow b = c + h$

$(i) \Rightarrow put\ x = a\ then\ a = c + y \Rightarrow c - h = c + y \Rightarrow -h = y$

$put\ x = b\ then\ c + h = c + y \Rightarrow h = y$

Now $\int_a^b f(x)dx = \int_{-h}^{+h} f(c+y)dy$ **where y is small change**

Using Taylor Series Formula $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots \ldots \ldots \ldots \ldots ..$

$$\int_{-h}^{+h} f(c+y)dy = \int_{-h}^{+h} \left[f(\,c) + yf'(c) + \frac{y^2}{2!}f''(c) + \cdots \ldots \ldots \ldots \ldots ..\right] dy$$

Neglecting higher derivatives

$$\int_{-h}^{+h} f(c+y)dy = \int_{-h}^{+h} \left[f(\,c) + yf'(c) + \frac{y^2}{2!}f''(c)\right] dy$$

$$\int_{-h}^{+h} f(c+y)dy = \left|yf(\,c) + \frac{y^2}{2}f'(c) + \frac{y^3}{2.3}f''(c)\right|_{-h}^{h} = 2h\left[f(c) + \frac{h^2}{6}f''(c)\right] \ldots \ldots \ldots \ldots \ldots \ldots .(i)$$

$$f(a) = f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!}f''(c) + neglected$$

$$f(b) = f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c) + neglected$$

$$f(c-h) - f(c+h) = 2f(c) + 2\frac{h^2}{2!}f''(c)$$

$f(c-h) - f(c+h) - 2f(c) = h^2 f''(c)$ **Put this value in (i)**

$$\int_a^b f(x)dx = 2h[f(c) + \frac{1}{6}\{f(c-h) + f(c+h) - 2f(c)\}]$$

$$\int_a^b f(x)dx = \frac{2h}{6}[6f(c) + f(c-h) + f(c+h) - 2f(c)]$$

$$\int_a^b f(x)dx = \frac{h}{3}[4f(c) + f(c-h) + f(c+h)] = \frac{h}{3}[4f(c) + f(a) + f(b)]$$

$$\int_a^b f(x)dx = \frac{h}{3}[4f(x_1) + f(x_0) + f(x_2)] = \frac{h}{3}[4y_1 + y_0 + y_2]$$

For n = 4

$$\int_{x_0}^{x_4} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$\int_{x_0}^{x_4} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3) + 2y_2 + y_4]$$

In General

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 \ldots \ldots .. y_{2N-1}) + 2(y_2 + y_4 \ldots \ldots + y_{2N-2}) + y_{2N}]$$

DERIVATION OF SIMPSON'S $(\frac{1}{3})$ RULE (2nd method)

$$\int_{x_0}^{x_{2N}} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots \ldots \ldots \ldots \ldots \ldots . \ldots \ldots + \int_{x_{2N-2}}^{x_{2N}} f(x)dx$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4] + \cdots \ldots \ldots \ldots \ldots \ldots .. + \frac{h}{3}[y_{2N-2} + 4y_{2N-1} + y_{2N}]$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 \ldots \ldots y_{2N-1}) + 2(y_2 + y_4 \ldots + y_{2N-2}) + y_{2N}]$$

This is required formula for Simpson's (1/3) Rule

EXAMPLE

Compute $I = \sqrt{\frac{2}{\pi}}\int_0^1 e^{-\frac{x^2}{2}}\mathrm{dx}$ **using Simpson's (1/3) Rule when** $h = 0.125$

SOLUTION

X	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1
F(x)	0.798	0.792	0.773	0.744	0.704	0.656	0.602	0.544	0.484

Since by Simpson's Rule

$$\sqrt{\frac{2}{\pi}}\int_0^1 e^{-\frac{x^2}{2}}\mathrm{dx} = \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$\sqrt{\frac{2}{\pi}}\int_0^1 e^{-\frac{x^2}{2}}\mathrm{dx} = 0.6827$ **After putting the values.**

SIMPSON'S $\left(\frac{3}{8}\right)$ RULE

Rule is based on fitting four points by a cubic.

Simpson's Rule is defined as for simple case

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] - \frac{3h^5}{80} y^{iv}(\S)$$

While in composite form ("n" must be divisible by 3) it is defined as

$$\int_{x_0}^{x_N} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + \cdots \ldots \ldots .. + y_{N-1}) + 2(y_3 + y_6 + \cdots \ldots \ldots .. + y_{N-3}) + y_N]$$

DERIVATION

$$\int_{x_0}^{x_N} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots \ldots \ldots \ldots \ldots . \ldots \ldots + \int_{x_{N-3}}^{x_N} f(x)dx$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] + \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

$$+ \cdots \ldots \ldots \ldots \ldots \ldots .. + \frac{3h}{8}[y_{N-3} + 3y_{N-2} + 3y_{N-1} + y_N]$$

$$\int_{x_0}^{x_N} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + \cdots \ldots \ldots .. + y_{N-1}) + 2(y_3 + y_6 + \cdots \ldots \ldots .. + y_{N-3}) + y_N]$$

This is required formula for Simpson's (3/8) Rule.

REMARK: Global error in Simpson's (1/3) and (3/8) rule are of the same order but if we consider the magnitude of error then Simpson (1/3) rule is superior to Simpson's (3/8) rule.

METHODS FOR NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

<u>SINGLE STEP METHODS</u>: A series for ‘y’ in terms of power of ‘x’ form which the value of ‘y’ at a particular value of ‘x’ can be obtained by direct substitution
e.g. Taylor’s, Picard’s, Euler’s, Modified Euler’s Method.

<u>MULTI - STEP METHODS</u>: In multi-step methods, the solution at any point ‘x’ is obtained using the solution at a number of previous points.
(Predictor- corrector method, Adam’s Moulton Method, Adam’s Bash forth Method)

REMARK

There are some ODE that cannot be solved using the standard methods. In such situations we apply numerical methods. These methods yield the solutions in one of two forms.

- **(i)** **A series for ‘y’ in terms of powers of ‘x’ from which the value of ‘y’ can be obtained by direct substitution. e.g. Taylor’s and Picard’s method**
- **(ii)** **A set of tabulated values of ‘x’ and ‘y’. e.g. and Euler’s, Runge Kutta**

ADVANTAGE/DISADVANTAGE OF MULTI - STEP METHODS

They are not self-starting. To overcome this problem, the single step method with some order of accuracy is used to determine the starting values.

Using these methods one step method clears after the first few steps.

LIMITATION (DISADVANTAGE) OF SINGLE STEP METHODS.

For one step method it is typical, for several functions evaluation to be needed.

IMPLICIT METHODS

Method that does not directly give a formula to the new approximation. A need to get it, need an implicit formula for new approximation in term of known data. These methods also known as close methods. It is possible to get stable 3rd order implicit method.

EXPLICIT METHODS

Methods that not directly give a formula to new approximation and need an explicit formula for new approximation "y_{i+1}" in terms of known data. These are also called open methods.

Most Authorities proclaim that it is not necessary to go to a higher order method. Explain.

Because the increased accuracy is offset by additional computational effort.

If more accuracy is required, then either a smaller step size. OR an adaptive method should be used.

CONSISTENT METHOD: A multi-step method is consistent if it has order at least one "1"

TAYLOR'S SERIES EXPANSION

Given (x) , smooth function. Expand it at point $x = c$ then

$$f(x) = f(c) + (x-c)f'(c) + \frac{(x-c)^2}{2!}f''(c) + \cdots \ldots \ldots \ldots$$

$$\Rightarrow f(c) = \sum_{k=0}^{\infty} \frac{(x-c)^k}{k!} f^k$$ **This is called Taylor's series of 'f' at 'c'**

If $x_0 - c = h \quad and \quad f(x) = y \quad then \Rightarrow c = x_0 + h$

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots$$

MECLAURIN SERIES FROM TAYLOR'S

If we put $c = 0$ in Taylor's series then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^2}{3!} f'''(0) + \cdots \ldots . \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^k(0)$$

ADVANTAGE OF TAYLOR'S SERIES

(1) One step, Explicit.
(2) Can be high order.
(3) Easy to show that global error is the same as local truncation error.
(4) Applicable to keep the error small.

DISADVANTAGE

Need to explicit form of the derivatives of function. That is why not practical.

ERROR IN TAYLOR'S SERIES

Assume $f^k(x)$ $(0 \leq k \leq n)$ are continuous functions. Call

$$f_n(x) = \sum_{k=0}^{n} \frac{(x-c)^k}{k!} f^k(c)$$ **Then first $(n+1)$ term is Taylor series**

Then the error is

$$E_{n+1} = f(x) - f_n(x) \sum_{k=n+1}^{\infty} \frac{(x-c)^k}{k!} f^k(c) = \frac{(x-c)^{n+1}}{(n+1)!} f^{n+1} (\S)$$

Where '$\S$' is some point between 'x' and 'c' .

CONVERGENCE

A Taylor's series converges rapidly if 'x' is nears 'c' and slowly (or not at all) if 'x' is for away form 'c'.

EXAMPLE

Obtain numerically the solution of $y' = t^2 + y^2$; $y(1) = 0$ using Taylor Series method to find 'y' at 1.3

SOLUTION

$y' = t^2 + y^2 \ldots \ldots \ldots (i)$

$y'' = 2t + 2yy' \ldots \ldots \ldots (ii)$ $\quad y''' = 2\left[1 + y'^2 + yy''\right] \ldots \ldots \ldots (iii)$

$y''' = 2[yy''' + 3y'y''] \ldots \ldots \ldots (iv)$ **...... ....and so on.**

where $y_0 = 0$ and $t_0 = 0$, $h = t - t_0 = 0.3$

therefore $(i) \Rightarrow y_0' = 1, (ii) \Rightarrow y_0'' = 2, (iii) \Rightarrow y_0''' = 4, (iv) \Rightarrow y_0^{iv} = 12$,.........................

Now by using formula $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2!} y''(t_0) + \cdots \ldots \ldots \ldots \ldots \ldots$
we get

$y(1 + 0.3) = y(1.3) = 0.4132$ as required.

SECOND ORDER RUNGE KUTTA METHOD

WORKING RULE: **For a given initial value problem of first order** $y' = f(x, y)$, $y(x_0) = y_0$

Suppose "$x_0, x_1, x_2 \ldots\ldots\ldots$" **be equally spaced 'x' values with interval 'h'**

i.e. $x_1 = x_0 + h$, $x_2 = x_1 + h$, $\ldots\ldots\ldots\ldots$

Also denote $y_0 = y(x_0)$, $y_1 = y(x_1)$, $y_2 = y(x_2) \ldots\ldots\ldots\ldots\ldots$

Then for "$n = 0, 1, 2 \ldots\ldots\ldots$" **until termination do:**

$x_{n+1} = x_n + h$,, $k_n = hf(x_n, y_n)$,, $I_n = hf(x_{n+1}, y_n + k_n)$

Then $y_{n+1} = y_n + \frac{1}{2}(k_n + I_n)$ **Is the formula for second order RK-method.**

REMARK: Modified Euler Method is a special case of second order RK-Method.

IN ANOTHER WAY: **If** $k_1 = hf(x_k, y_k)$, $k_2 = hf(x_{k+1}, y_k + k_1)$

Then Equation for second order method is $y_{k+1} = y_k + \frac{1}{2}(k_1 + k_2)$

This is called <u>Heun's Method</u>

ANOTHER FORMULA FOR SECOND ORDER RK-METHOD

$y_{n+1} = y_n + \frac{1}{3}(2k_1 + k_2)$ **Where** $k_1 = hf(t_n, y_n)$, $k_2 = hf\left(t_n + \frac{3}{2}h,\ y_n + \frac{3}{2}k_1\right)$

LOCAL TRUNCATION ERROR IN RK-METHOD.

LTE in RK-method is the error that arises in each step simply because of the truncated Taylor series. This error is inevitable. Error of Runge Kutta method of order two involves an error of O(h^3).

In General RK-method of order 'm' takes the form $x_{k+1} = x_k + w_1k_1 + w_2k_2 + \cdots + w_mk_m$

Where $k_1 = h.f(t_k, x_k)$, $k_2 = hf(t_k + a_2h, x + b_2k_1)$

$k_3 = h\,f(t_k + a_3h, x + b_3k_1 + c_3k_2)$ ……………………..$k_m = h.f\left(t_k + a_mh, x + \sum_{i=1}^{m-1} \emptyset_i\, k_i\right)$

MULTI STEP METHODS OVER RK-METHOD (PREFRENCE): **Determination of** y_{i+1} **require only on evaluation of** $f\ (t, y)$ **per step. Whereas RK-method for** $n \geq 3$ **require four or more function evaluations. For this reason, multi-step methods can be twice as fast as RK-method of comparable Accuracy.**

EXAMPLE: use second order RK method to solve $\frac{dy}{dx} = \frac{y+x}{y-x} = f(x,y)\,;\, y(0) = 1$ at x =0.4 and h=0.2

SOLUTION: $\frac{dy}{dx} = \frac{y+x}{y-x} = f(x,y) \quad \dots\dots\dots\dots\dots\dots.(i)$

If 'h' is not given then use by own choice for 4 – step take h=0.1 and for 1 – step take h=0.4

Given that h=0.2 , $x_0 = 0, x_1 = x_0 + h = 0.2, x_2 = 0.4$

Now using formula of order two

$y_{n+1} = y_n + \frac{1}{3}(2k_1 + k_2)$ **Where** $k_1 = hf(x_n, y_n)$, $k_2 = hf\left(x_n + \frac{3}{2}h,\ y_n + \frac{3}{2}k_1\right)$

$k_1 = hf(x_0, y_0) = 0.2,\quad k_2 = hf\left(x_n + \frac{3}{2}h, y_n + \frac{3}{2}k_1\right) = 0.32$

$\therefore n = \frac{x - x_0}{h}$

n = 2 steps

For n = 0; $k_1 = hf(x_0, y_0) = 0.2,\quad k_2 = hf\left(x_0 + \frac{3}{2}h, y_0 + \frac{3}{2}k_1\right) = 0.32$

$(i) \Rightarrow y_1 = y_0 + \frac{1}{3}(2k_1 + k_2) = 1.24 \Rightarrow y(0.2) = 1.24$

For n = 1; $k_1 = hf(x_1, y_1) = 0.2769,\quad k_2 = hf\left(x_1 + \frac{3}{2}h, y_1 + \frac{3}{2}k_1\right) = 0.3731$

$(i) \Rightarrow y_2 = y_1 + \frac{1}{3}(2k_1 + k_2) = 1.54897 \Rightarrow y(0.4) = 1.54897$

CLASSICAL RUNGE KUTTA METHOD (RK – METHOD OF ORDER FOUR)

ALORITHM: **Given the initial value problem of first order** $y' = f(x,y)$, $y(x_0) = y_0$

Suppose "$x_0, x_1, x_2 \dots\dots\dots$" **be equally spaced 'x' values with interval 'h'**

i.e. $x_1 = x_0 + h$, $x_2 = x_1 + h$, $\dots\dots\dots\dots$

Also denote $y_0 = y(x_0),\quad y_1 = y(x_1),\quad y_2 = y(x_2) \dots\dots\dots\dots\dots$

Then for "$n = 0, 1, 2 \dots\dots\dots..$" **until termination do:**

$x_{n+1} = x_n + h$,, $k_1 = hf(x_n, y_n)$ $\quad k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \quad k_4 = hf(x_n + h, y_n + k_3)$

Then $\quad y_{n+1} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_n$

Is the formula for Runge Kutta method of order four and its error is "$O(h^5)$"

ADVANTAGE OF METHOD

- **Accurate method.**
- **Easy to compute for the use of computer.**
- **It lakes in estimating the error.**
- **Easy to program and is efficient.**

COMPUTATIONAL COMPARISON: **The main computational effort in applying the Runge Kutta method is the evaluation of 'f'. In RK – 2 the cost is two function evaluation per step. In RK – 4 require four evaluations per step.**

EXAMPLE: use 4th order RK method to solve $\frac{dy}{dx} = t + y; y(0) = 1$ from t =0 to 0.4 **taking h = 0.4**

SOLUTION: $\frac{dy}{dx} = t + y \quad \ldots\ldots\ldots\ldots\ldots\ldots.(i)$

$h = 0.1, t_0 = 0, t_1 = t_0 + h = 0.1 \quad , \quad t_2 = 0.2 \; , t_3 = 0.3, t_4 = 0.4$

Now using formulas for the RK method of 4th order

$$y_{n+1} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Where $k_1 = hf(t_n, y_n)$, $k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$, $k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$ $k_4 = hf(t_n + h, y_n + k_3)$

STEP I : for n=0;

$k_1 = hf(t_0, y_0) = 0.1$, $k_2 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.11$

$k_3 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1105 \quad k_4 = hf(t_0 + h, y_0 + k_3) = 0.12105$

$(ii) \Rightarrow y_1 = y(0.1) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_0 = 1.11034$

STEP II : for n=1;

$k_1 = hf(t_1, y_1) = 0.121034$, $k_2 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.13208$

$k_3 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.132638 \quad k_4 = hf(t_1 + h, y_1 + k_3) = 0.1442978$

$(ii) \Rightarrow y_1 = y(0.2) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_1 = 1.2428$

STEP III : for n=2;

$k_1 = hf(t_2, y_2) = 0.14428$, $k_2 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.156494$

$k_3 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1571047 \quad k_4 = hf(t_2 + h, y_2 + k_3) = 0.16999047$

$(ii) \Rightarrow y_1 = y(0.3) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_2 = 1.399711$

THIS IS REQUIRED ANSWER

PREDICTOR - CORRECTOR METHODS

A predictor corrector method refers to the use of the predictor equation with one subsequent application of the corrector equation and the values so obtained are the final solution at the grid point.

PREDICTOR FORMULA

The explicit (open) formula used to predict approximation "y_{i+1}^{n}"is called a predictor formula.

CORRECTOR FORMULA

The implicit (closed) formula used to determine "y_{i+1}^{n}"is called Corrector Formula. This used to improve "y_{i+1}"

IN GENERAL

Explicit and Implicit formula are used as pair of formulas. The explicit formula is called 'predictor' and implicit formula is called 'corrector'

Implicit methods are often used as 'corrector' and Explicit methods are used as 'predictor' in predictor-corrector method. why?

Because the corresponding Local Truncation Error formula is smaller for implicit method on the other hand the implicit methods has the inherent difficulty that extra processing is necessary to evaluate implicit part.

REMARK

- **Truncation Error of predictor is** $E_p = \frac{14}{45} h^5 y_{k-1}^{(5)} \quad OR \quad \frac{28}{90} h\Delta^4 y_0'$
- **Local Truncation Error of Adam's Predictor is** $\frac{251}{720} h^5 y^{(5)}$
- **Truncation Error of Corrector is** $\frac{1}{90} h\Delta^4 y_0'$

Why Should one bother using the predictor corrector method When the Single step method are of the comparable accuracy to the predictor corrector methods are of the same order?

A practical answer to that relies in the actual number of functional evaluations. For example, RK - Method of order four, each step requires four evaluations where the Adams Moulton method of the same order requires only as few as two evaluations. For this reason, predictor corrector formulas are in General considerably more accurate and faster than single step methods.

REMEMBER

In predictor corrector method if values of "$y_0, y_1, y_2 \ldots \ldots \ldots$" against the values of "$x_0, x_1, x_2 \ldots \ldots \ldots$" are given the we use symbol predictor corrector method and in this method we use given values of"$y_0, y_1, y_2 \ldots \ldots \ldots$"

If "$y_0, y_1, y_2 \ldots \ldots \ldots$" Are not given against the values of "$x_0, x_1, x_2 \ldots \ldots \ldots$" then we first find values of "$y_0, y_1, y_2 \ldots \ldots \ldots$" by using RK - method
OR By using formula $\forall\, j = 1, 2, 3 \ldots \ldots \ldots n$

$$y_j = y_0 + (jh)y_0' + \frac{(jh)^2}{2!} y_0'' + \frac{(jh)^3}{3!} y_0''' + \ldots \ldots \ldots \ldots ..$$

BASE (MAIN IDEA) OF PREDICTOR CORRECTOR METHOS

In predictor corrector methods a predictor formula is used to predict the value of 'y' at t_{n+1} and then a corrector formula is used to improve the value of y_{n+1}
Following are predictor – corrector methods

1. **Milne's Method**
2. **Adam – Moulton method**

MILNE'S METHOD

It's a multi-step method. In General, Milne's Predictor – Corrector pair can be written as

P: $y_{n+1} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n') \quad n \geq 3$

C: $y_{n+1} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}') \quad n \geq 3$

REMARK: Magnitude of truncation error in Milne's corrector formula is $\frac{1}{90} h\, \Delta^4 y_0'$ and truncation error in Milne's predictor formula is $\frac{28}{90} h\, \Delta^4 y_0'$
stable, convergent, efficient, accurate, compeer friendly.

ALGORITHM

- **First predict the value of y_{n+1} by above predictor formula.**
  Where derivatives are computed using the given differential equation itself.
- **Using the predicted value "y_{n+1}" we calculate the derivative $y_{n+1}'$from the given differential Equation.**
- **Then use the corrector formula given above for corrected value of y_{n+1}. Repeat this process.**

EXAMPLE: use Milne's method to solve $\frac{dy}{dx} = 1 + y^2$; $y(0) = 0$ **and compute y(0.8)**

SOLUTION: $h = 0.2, x_0 = 0, x_1 = x_0 + h = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$ *also* $y_0 = 0$

Now by using Euler's method $\Rightarrow y_{m+1} = y_m + hf(t_m, y_m)$

$for\ m = 0; \quad \Rightarrow y_1 = y_0 + hf(t_0, y_0) = 0.2 = y(0.2)$

$for\ m = 1; \quad \Rightarrow y_2 = y_1 + hf(t_1, y_1) = 0.48 = y(0.4)$

$for\ m = 2; \quad \Rightarrow y_3 = y_2 + hf(t_2, y_2) = 0.73 = y(0.6)$

Now $y'_n = 1 + y_n^2$

For n=1 $\Rightarrow y'_1 = 1 + y_1^2 = 1.04$

For n=2 $\Rightarrow y'_2 = 1 + y_2^2 = 1.16$

For n=3 $\Rightarrow y'_3 = 1 + y_3^2 = 1.36$

Now using Milne's Predictor formula

P: $y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n) \quad n \geq 3$

$y_4 = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) = 0.98 \Rightarrow y'_4 = 1 + y_4^2 = 1.9604$

Now using corrector formula

C: $y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) \quad n \geq 3$

$y_4 = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) = 1.05 = y(0.8)$